Yet Another Hierarchy Theorem

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Abstract

$n+1$ nested k-ary fixed point operators are more expressive than n. This holds on finite structures for practically all fixed point logics.

1 Introduction

Fixed point logics are extensions of first order logic by fixed point operators, which allow inductive definitions. They have turned out to be of importance in finite model theory
In most fixed point logics several nested fixed point operators can be collapsed to one without losing expressiveness. This is achieved by increasing the arity of the fixed point operator (i.e the arity of the second order induction variable). On the other hand, Grohe has shown in [Gro96] that $k+1$-ary fixed point operators are more expressive than k-ary fixed point operators. This lead to the conjecture that nesting k-ary fixed point operators increases the expressive power.
In the present paper we prove that this conjecture is true for all k and all reasonable logics between deterministic transitive closure logic DTC and partial fixed point logic PFP. We show this on a class of finite unordered structures of a fixed signature with a ternary relation.
The result also holds on finite graphs, but the graph reduction contains some tricky bits and is not really worth the effort.
On ordered structures, the above conjecture is still open for most cases. If it fails for least fixed point logic LFP and some $k \geq 2$, this would imply that LOGSPACE$\subset$TIME(n^m) for some $m \in I\!N$. I am not aware of any case in which showing the conjecture on ordered structures would have implications in complexity theory of similar importance.
Flum and Grohe observed ([FG97]) that the results of Mats and Thomas in [MT??] imply that the above conjecture is true on ordered structures for $k = 1$ and all logics between least fixed point logic LFP and monadic second order logic.

2 The Result

Some conventions: $\mathsf{Univ}(\mathcal{A})$ denotes the universe of the structure $\mathcal{A}$, $\mathsf{Aut}(\mathcal{A})$ its automorphism group. Deviating from common practice, we use α, β, γ for elements of a structure.
For a second order variable X, ar(X) is its arity. Whenever we write $X\bar{t}$, $R\bar{t}$ or $[\mathrm{PFP}_{X\bar{x}}\ \varphi]\bar{t}$, we assume that the arity of X or R, respectively, coincides with the length of the tuple of terms $\bar{t}$ (and the tuple of variables $\bar{x}$).

Definition 1 (PFP syntax) For a signature σ the set of PFP[σ] formulas is given by the following calculus:

$$\frac{}{t_1 \doteq t_2}; \quad \frac{}{X\bar{t}}; \quad \frac{}{R\bar{t}}, R\in\sigma; \quad \frac{\varphi}{\neg\varphi}; \quad \frac{\varphi,\psi}{(\varphi\wedge\psi)}; \quad \frac{\varphi}{\exists x\varphi}; \quad \frac{\varphi}{[\mathrm{PFP}_{X\bar{x}}\ \varphi]\bar{t}} \ . \qquad \square$$

Definition 2 (PFP semantics) The relation $\mathcal{A} \models \psi$ is defined as usual by induction on ψ. In the main step for $\psi = [\mathrm{PFP}_{X\bar{x}}\ \varphi(X,\bar{x})]\bar{v}$ we set

$$X_0^{\mathcal{A}} := \emptyset, \qquad X_{i+1}^{\mathcal{A}} := \{\bar{\alpha} \in \mathcal{A} \mid \mathcal{A} \models \varphi(X_i^{\mathcal{A}}, \bar{\alpha})\}.$$

If there is a $k \in I\!N$ with $X_{k+1}^{\mathcal{A}} = X_k^{\mathcal{A}}$, we say $X_\infty^{\mathcal{A}}$ exists and set $X_\infty^{\mathcal{A}} := X_k^{\mathcal{A}}$. Finally, we define:

$$\mathcal{A} \models [\mathrm{PFP}_{X\bar{x}}\ \varphi(X,\bar{x})]\bar{\alpha} \ :\Leftrightarrow\ X_\infty^{\mathcal{A}} \text{ exists, and } \bar{\alpha} \in X_\infty^{\mathcal{A}}$$

$\square$

Definition 3 (DTC(E)) Let E be a binary relation on a set M. The *deterministic transitive closure* DTC(E) of R is defined by
DTC(E) $:=$ $\{(\alpha,\beta) \in M \mid$ there exist $n > 0$ and $e_0,\ldots,e_n \in M$ such that
$\alpha = e_0, \beta = e_n$, and e_{i+1} is the *unique* e with $(e_i, e_{i+1}) \in E$ for all $i < n\}$. $\square$

Deterministic transitive closure logic DTC is first order logic extended by operators for the deterministic transitive closure of definable $2k$-ary relations. The Hierarchy Theorem below implies a hierarchy theorem for all logics between DTC and PFP. In the proof we only need a very small subset of DTC. We do not use $\doteq$, $\neg$, $\vee$ or $\forall$ and we use DTC operators only on binary relations. To make the Hierarchy Theorem as strong as possible, we explicitly define this subset and call it the *restricted deterministic transitive closure logic* RDTC.

Definition 4 (RDTC[σ] syntax) The set of RDTC[σ] formulas is given by the following calculus: $\frac{}{R\bar{t}}, R\in\sigma; \quad \frac{\varphi,\psi}{(\varphi\wedge\psi)}; \quad \frac{\varphi}{\exists x\varphi}; \quad \frac{\varphi}{[\mathrm{DTC}_{xy}\ \varphi]st}.$ $\square$

Definition 5 (RDTC semantics)

$$\mathcal{A} \models [\mathrm{DTC}_{xy}\ \varphi(x,y)]\alpha\beta \quad :\Leftrightarrow \quad (\alpha,\beta) \in \mathrm{DTC}(\ \{(\alpha',\beta') \mid \mathcal{A} \models \varphi(\alpha',\beta')\}\ )$$

$\square$

An approach in formalizing the nesting hierarchy would be to restrict the arity of all fixed point operators to some k and then to consider the nesting number. We take a different way and define a rank function Qr on all fixed point formulas that refines the hierarchy of nesting numbers for all arities k. Qr sums up the arities of nested fixed point operators. Since Qr only counts second order variables, we call it the *second order rank.*

Definition 6 (Qr) The *second order rank* Qr is defined on all PFP and RDTC formulas by

$$\begin{array}{lcl}
\mathrm{Qr}(\psi) & := & 0, \text{ if } \psi \text{ is an atom} \\
\mathrm{Qr}(\neg\varphi) & := & \mathrm{Qr}(\varphi) \\
\mathrm{Qr}(\varphi \wedge \chi) & := & \max(\mathrm{Qr}(\varphi), \mathrm{Qr}(\chi)) \\
\mathrm{Qr}(\exists x\varphi) & := & \mathrm{Qr}(\varphi) \\
\mathrm{Qr}([\mathrm{PFP}_{X\bar{x}}\ \varphi]\bar{t}) & := & \mathrm{Qr}(\varphi) + \mathrm{ar}(X) \\
\mathrm{Qr}([\mathrm{DTC}_{xy}\ \varphi]st) & := & \mathrm{Qr}(\varphi) + 1
\end{array}$$

$\square$

Let us now fix the signature. We set $\sigma_0 := \{R, a, b\}$, where R is a ternary relation symbol and a, b are constants.

We are now in a position to write down the full result of this paper. The purpose of this paper is to prove the following Hierarchy Theorem:

Theorem 7 (Hierarchy Theorem) *For each $n \in \mathbb{N}$ there is an* RDTC$[\sigma_0]$-*sentence of second order rank $n+1$, which on finite structures is not equivalent to any* PFP *sentence of second order rank $\leq n$.*

PFP strongly contains RDTC in the sense that every RDTC formula is equivalent to a PFP formula of the same second order rank in which only unary fixed point operators occur. Therefore the result which was announced in the introduction is a corollary to the Hierarchy Theorem:

Corollary 8 *$n+1$ nested k-ary fixed point operators are more expressive than n.*

This holds for all sublogics of PFP that strongly contain RDTC in the above sense, most notably PFP, IFP, LFP, SFP, TC, DTC. For the definitions and whereabouts of these logics see [EF95].

Remark 9 If we delete the words "e_{i+1} is the unique e with" in definition 3, it becomes a definition of the *transitive closure* TC(E). All DTC operators in this paper can be replaced by TC operators without harm. Let RTC be analogue of RDTC for the transitive closure operator. Then the above corollary also holds for all sublogics of PFP that strongly contain RTC. An example for this are the existential fragments of LFP and TC. Note that RTC does not contain RDTC (RTC formulas are preserved under embeddings, but the deterministic transitive closure is not.).

Remark 10 The Hierarchy Theorem also holds for simultaneous partial fixed point logic S-PFP instead of PFP, if we define
$\mathrm{Qr}([\text{S-PFP}_{X_0\bar{x}_0,\ldots,X_m\bar{x}_m}\ \varphi_0,\ldots,\varphi_m]\bar{t}) := \max\{\mathrm{Qr}(\varphi_i)\}+\max\{\mathrm{Qr}(X_i)\}$. Extending the proofs given here to S-PFP involves no new ideas, but some notational overhead.

Also for information on S-PFP, see [EF95].

3 The Game

To prove that certain structures cannot be distinguished by formulas of second order rank n, we will define an Ehrenfeucht-Fraïssé type pebble game. As usual, in this game each pebble corresponds to a first order variable. Hence the second order rank alone is not sufficient to define the parameters for the game. Therefore we additionally define a *first order rank* qr, which counts first order variables.

Definition 11 (qr) The *first order rank* qr is defined on all PFP formulas by:

$$\begin{array}{lcll}
\mathrm{qr}(\psi) & := & 0, \text{ if } \psi \text{ is an atom} & \\
\mathrm{qr}(\neg\varphi) & := & \mathrm{qr}(\varphi) & \\
\mathrm{qr}(\varphi \wedge \chi) & := & \max(\mathrm{qr}(\varphi), \mathrm{qr}(\chi)) & \\
\mathrm{qr}(\exists x\varphi) & := & \mathrm{qr}(\varphi)+1 & \text{\% Here is the difference to Qr} \\
\mathrm{qr}([\mathrm{PFP}_{X\bar{x}}\ \varphi]\bar{t}) & := & \mathrm{qr}(\varphi)+\mathrm{ar}(X) &
\end{array}$$

□

We now define the game $\mathrm{G}_0(k,n,\mathcal{A},\mathcal{B})$ in such a way that if the duplicator wins[1] $\mathrm{G}_0(k,n,\mathcal{A},\mathcal{B})$, then $\mathcal{A} \models \psi \Leftrightarrow \mathcal{B} \models \psi$ for all PFP sentences ψ with $\mathrm{qr}(\psi) \leq k$ and $\mathrm{Qr}(\psi) \leq n$.
It is derived from a game Grohe uses in [Gro96].

Definition 12 ($\mathrm{G}_0(k,n,\mathcal{A},\mathcal{B})$) For structures $\mathcal{A}$ and $\mathcal{B}$ the game $\mathrm{G}_0(k,n,\mathcal{A},\mathcal{B})$ is played by two players on $\mathcal{A}$ and $\mathcal{B}$ with $2k$ pebbles $P_1, Q_1 \ldots P_k, Q_k$.
We say that a pair (P_i, Q_i) of pebbles is on the board in a situation of the game if it is placed on the structures in that situation. The other pebbles are called *free*. Each situation of the game and each pebble on the board has a *depth* ≥ 0. Each depth lower or equal to the depth of the situation has an *arity* ≥ 0. The game starts in the situation with depth 0 and all pebbles free. The arity of depth 0 is defined to be 0. In each situation of the game the challenger selects one of the following moves:

$\exists$-move: The challenger places a free pebble P_i on an element $\alpha_i \in \mathcal{A}$. The duplicator places the corresponding pebble Q_i on an element $\beta_i \in \mathcal{B}$. The depth of P_i and Q_i is defined to be the current depth of the game.

$\forall$-move: The challenger places a free pebble Q_i on an element $\beta_i \in \mathcal{B}$. The duplicator places the corresponding pebble P_i on an element $\alpha_i \in \mathcal{A}$. The depth of P_i and Q_i is defined to be the current depth of the game.

I-move: The depth of the game is increased by one. The challenger assigns an arity $\mathrm{ar}(d) > 0$ to the new depth d of the game, such a way that $\sum_{0 \leq e \leq d} \mathrm{ar}(e) \leq n$.

R-move: The challenger 'reduces'[2] the depth d' of the game to some $d \leq d'$ with $\mathrm{ar}(d) > 0$. Then she selects $\mathrm{ar}(d)$ pairs (P_i, Q_i) of depth $\geq d$ to be left on the board. All other pebbles of depth $\geq d$ are removed from the structures.

In each situation the pairs of pebbled elements (α_i, β_i) and the pairs of constants $(c^{\mathcal{A}}, c^{\mathcal{B}})$ are called *couples*. The duplicator wins the play if in each situation the couples form a partial isomorphism from $\mathcal{A}$ to $\mathcal{B}$. □

Theorem 13 (Game Theorem) *Let σ be a signature without function symbols. Suppose that $\mathcal{A}, \mathcal{B}$ σ-structures such that the duplicator wins $\mathrm{G}_0(k,n,\mathcal{A},\mathcal{B})$. Then $\mathcal{A} \models \psi \Leftrightarrow \mathcal{B} \models \psi$ holds for all PFP sentences ψ with $\mathrm{qr}(\psi) \leq k$ and $\mathrm{Qr}(\psi) \leq n$.*

Proof: We prove for every PFP$[\sigma]$ sentence ψ:

$(*)$ If $\mathcal{A} \models \psi$ and $\mathcal{B} \not\models \psi$ then the challenger has a winning strategy in every situation of depth d, in which $\mathrm{Qr}(\psi) + \sum_{0 \leq e \leq d} \mathrm{ar}(e) \leq n$ and at least $\mathrm{qr}(\psi)$ pebbles are free.

As the definition of the game is symmetric in $\mathcal{A}$ and $\mathcal{B}$, the Game Theorem follows immediately from $(*)$. We prove $(*)$ by induction on ψ, in either case advising the challenger how to win:

If ψ is an atom: ψ is an atomic sentence, so all terms occurring in ψ are constants. As ψ holds in $\mathcal{A}$ and not in $\mathcal{B}$, the mapping between the constants of the respective structures is not a partial isomorphism. The duplicator has already lost in the start situation.

[1] For a game G by "X wins G" we mean "X has a winning strategy for G".
[2] The quotation remind of the fact that we do not exclude $d = d'$.

If $\psi = \neg\varphi$: ψ holds in $\mathcal{A}$ but not in $\mathcal{B}$, so φ holds in $\mathcal{B}$ but not in $\mathcal{A}$. The induction hypothesis gives a winning strategy for $\mathrm{G}_0(\mathrm{qr}(\varphi), \mathrm{Qr}(\varphi), \mathcal{B}, \mathcal{A})$. Use this winning strategy for φ with interchanged roles of $\mathcal{A}$ and $\mathcal{B}$ ($\exists$-moves become $\forall$-moves and vice versa).

If $\psi = \varphi \wedge \chi$: φ and χ both hold in $\mathcal{A}$, one of them does not hold in $\mathcal{B}$. For the latter use the winning strategy given by the induction hypothesis.

If $\psi = \exists x \varphi(x)$: Select an $\exists$-move and place some pebble P_i on some $\alpha_i \in A$ for which $\varphi(\alpha_i)$ holds in $\mathcal{A}$. The duplicator has to place Q_i on an $\beta_i \in \mathcal{B}$. $\varphi(\beta_i)$ does not hold in $\mathcal{B}$ because otherwise ψ would. For the rest of the game regard x as a constant, being interpreted in $\mathcal{A}$ by α_i and in $\mathcal{B}$ by β_i. Use the winning strategy that is given by the induction hypothesis for φ, since there are still $\mathrm{qr}(\psi) - 1 = \mathrm{qr}(\varphi)$ pebbles free and $\mathrm{Qr}(\psi) = \mathrm{Qr}(\varphi)$.

If $\psi = [\mathrm{PFP}_{X\bar{x}}\ \varphi(\bar{x}, X)]\bar{t}$:
Define $X_i^{\mathcal{A}}$ and $X_\infty^{\mathcal{A}}$ as in definition 2. $X_\infty^{\mathcal{A}}$ exists, because $\mathcal{A} \models \psi$.

Case I: $X_\infty^{\mathcal{B}}$ also exists.
As ψ is a sentence, all components of $\bar{t}$ are constants. Since ψ holds in $\mathcal{A}$ but not in $\mathcal{B}$, there is i with $\bar{t}^{\mathcal{A}} \in X_i^{\mathcal{A}}$, but $\bar{t}^{\mathcal{B}} \notin X_i^{\mathcal{B}}$. Now do an I-move, choose the arity of the new depth to be $\mathrm{ar}(X)$ and the claim $(**)$ below guarantees you a winning strategy.

Case II: $X_\infty^{\mathcal{B}}$ does not exist.
Do l $\forall$-moves, where l is the length of $\bar{x}$. Place $Q_1, \ldots, Q_l$ to $\bar{\alpha} \in \mathcal{A}$ such that $\bar{\alpha} \in X_i^{\mathcal{B}}$ for infinitely many i and $\bar{\alpha} \notin X_i^{\mathcal{B}}$ for infinitely many i. Then the duplicator places $P_i, \ldots, P_l$ to some $\bar{\beta} \in \mathcal{B}$. If $\bar{\alpha} \in X_\infty^{\mathcal{A}}$, there is an i with $\bar{\alpha} \in X_i^{\mathcal{A}}, \bar{\beta} \notin X_i^{\mathcal{B}}$. Then as in case I use $(**)$, doing an I-move. If $\bar{\alpha} \notin X_\infty^{\mathcal{A}}$, then there is an i with $\bar{\alpha} \notin X_i^{\mathcal{A}}, \bar{\beta} \in X_i^{\mathcal{B}}$. Swap the roles of $\mathcal{A}, \mathcal{B}$ and also use $(**)$.

$(**)$ For all $i \in \mathbb{N}$ the challenger has a winning strategy in a situation with depth d, if:

(i) There are couples $(\alpha_1, \beta_1), \ldots, (\alpha_l, \beta_l)$, such that $\bar{\alpha} \in X_i^{\mathcal{A}}$ and $\bar{\beta} \notin X_i^{\mathcal{B}}$.

(ii) $\mathrm{ar}(d) \geq \mathrm{ar}(X)$, at least $\mathrm{qr}(\varphi)$ pebbles are free and $\sum_{0 \leq e \leq d} \mathrm{ar}(e) + \mathrm{Qr}(\varphi) \leq n$.

Proof of $(**)$ via induction on i:
For $i = 0$ there is nothing to show, since $X_0^{\mathcal{A}} = \emptyset$. Now let $i > 0$ and the situation of $(**)$ be given for $\bar{\alpha}, \bar{\beta}$. φ holds in $\mathcal{A}' := (\mathcal{A}, \bar{\alpha}, X_{i-1}^{\mathcal{A}})$ whereas it does not hold in $\mathcal{B}' := (\mathcal{B}, \bar{\beta}, X_{i-1}^{\mathcal{B}})$. Play according to the winning strategy for $G := \mathrm{G}_0(\mathrm{qr}(\varphi), \mathrm{Qr}(\varphi), \mathcal{A}', \mathcal{B}')$, which is given for the current situation by the induction hypothesis for φ.

Case 1 You win at the end of G because there are couples $(\alpha_1', \beta_1') \ldots (\alpha_l', \beta_l')$ such that $\bar{\alpha}' \in X_{i-1}^{\mathcal{A}}$ and $\bar{\beta}' \notin X_{i-1}^{\mathcal{B}}$:
In this case do an R-move, reducing the depth to d and selecting $\bar{\alpha}'$ and $\bar{\beta}'$ to be left on the board (as far as they are not constants anyway). The situation that is produced by this R-move matches (i) and (ii) for $i - 1$.

Case 2: You win at the end of G because the couples (α_i', β_i') do not form a partial isomorphism from $\mathcal{A}'$ to $\mathcal{B}'$ for other reasons: This means that you've already won the original game.

$\blacksquare$

In the definition of $G_0(n, k, \mathcal{A}, \mathcal{B})$, defining $ar(0) := 0$ seems somewhat deliberate. Indeed, in this point we need more sophistication when proving the Hierarchy Theorem.

Definition 14 $(G_m(n, k, \mathcal{A}, \mathcal{B}), G_*(n, k, \mathcal{A}, \mathcal{B}))$
For $m > 0$, the game $G_m(n, k, \mathcal{A}, \mathcal{B})$ is played like $G_0(n, k, \mathcal{A}, \mathcal{B})$, except that the arity of depth 0 is defined to be m instead of 0.
The game $G_*(n, k, \mathcal{A}, \mathcal{B})$ is played like $G_0(n, k, \mathcal{A}, \mathcal{B})$, except that the challenger is obliged to start with an I-move.
For both of the above games in case $n = 0$, the duplicator is defined to win the game iff the couples form a partial isomorphism in the start situation. $\square$

Proposition 15 *The duplicator wins* $G_0(n, k, \mathcal{A}, \mathcal{B})$ *iff for all* $0 \leq m \leq n$ *she wins* $G_m(n, k, \mathcal{A}, \mathcal{B})$.

Proposition 16 *The duplicator wins* $G_*(n, k, \mathcal{A}, \mathcal{B})$ *iff for all* $0 < m \leq n$ *she wins* $G_m(n, k, \mathcal{A}, \mathcal{B})$.

4 Assorted Sequences

To prove the Hierarchy Theorem, for every $n \in \mathbb{N}$ we have to construct pairs of σ_0-structures which are similar with respect to the above game, but can be distinguished by RDTC formulas of second order rank $n+1$. These structures will be complicated looking expansions of the now to be defined *assorted sequences*.

Definition 17 (assorted sequence) $Y = (\mathsf{Set}, <, \mathsf{Part})$ is an *assorted sequence* if Set is a finite set, $<$ a total order on Set and Part a partition of Set (i.e. $\bigcup \mathsf{Part} = \mathsf{Set}$ and $\forall L, M \in \mathsf{Part} : L \cap M = \emptyset$). $\square$

In the sequel $Y = (\mathsf{Set}, <, \mathsf{Part})$ is an arbitrary assorted sequence. The following definitions are relative to Y.

Definition 18 (N) $\mathsf{N} := |\mathsf{Set}|$ $\square$

Definition 19 ($\underline{i}$) $\underline{i}$ denotes the i-th element of $(\mathsf{Set}, <)$ $\square$

Thus $\underline{0}$ is the first, $\underline{\mathsf{N}-1}$ the last element of $(\mathsf{Set}, <)$.

Definition 20 ($\mathsf{Vert}, \underline{i}_s$) $\mathsf{Vert} := (\mathsf{Set} \cup \{\underline{\mathsf{N}}\}) \times \{\mathsf{left}, \mathsf{right}\}$, where $\underline{\mathsf{N}}$ is some individual which is not an element of Set. For $\underline{i} \in \mathsf{Set} \cup \{\underline{\mathsf{N}}\}, s \in \{\mathsf{left}, \mathsf{right}\}$, we abbreviate $\underline{i}_s := (\underline{i}, s)$. The elements of Vert are called *vertices* of Y. $\square$

Definition 21 (mirror) $\mathsf{mirror} : \mathsf{Vert} \to \mathsf{Vert}$ is defined by $\mathsf{mirror}(\underline{i}_{\mathsf{left}}) := \underline{i}_{\mathsf{right}}$ and $\mathsf{mirror}(\underline{i}_{\mathsf{right}}) := \underline{i}_{\mathsf{left}}$ $\square$

Definition 22 (connector, Conn^M**)** f is a *connector* for $M \in \mathsf{Part}$ if $f : M \to \{\|, \times\}$ and the number of elements of M that are mapped to $\times$ is even. Conn^M is the set of connectors for M. $\square$

A connector is also a partial function on Set.

Definition 23 (Conn) $\mathsf{Conn} := \bigcup_{M \in \mathsf{Part}} \mathsf{Conn}^M$ $\square$

Definition 24 ($\mathcal{Y}$) $\quad \mathcal{Y}$ is the $\{R\}$-structure defined by
$\mathsf{Univ}(\mathcal{Y}) := \mathsf{Vert} \cup \mathsf{Conn}$ and

$$\begin{aligned} \alpha\beta\gamma \in R^{\mathcal{Y}} :\Leftrightarrow \quad & \alpha = \underline{i}_s, \beta = \underline{i+1}_t \text{ for some } \underline{i} \in \mathsf{Set},\ s,t \in \{\mathsf{left}, \mathsf{right}\}, \\ & \gamma \in \mathsf{Conn} \text{ with } \underline{i} \in \mathrm{def}(\gamma) \\ & \text{and either } \ \gamma(\underline{i}) = \| \wedge s = t \ \text{ or } \ \gamma(\underline{i}) = \times \wedge s \neq t \end{aligned}$$

□

Definition 25 ($\mathcal{Y}^{\|}$) $\quad \mathcal{Y}^{\|}$ is the σ_0-extension of $\mathcal{Y}$ that is defined by
$a^{\mathcal{Y}^{\|}} := \underline{0}_{\mathsf{right}}$, $b^{\mathcal{Y}^{\|}} := \underline{\mathsf{N}}_{\mathsf{right}}$ □

Definition 26 ($\mathcal{Y}^{\times}$) $\quad \mathcal{Y}^{\times}$ is the σ_0-extension of $\mathcal{Y}$ that is defined by
$a^{\mathcal{Y}^{\times}} := \underline{0}_{\mathsf{right}}$, $b^{\mathcal{Y}^{\times}} := \underline{\mathsf{N}}_{\mathsf{left}}$ □

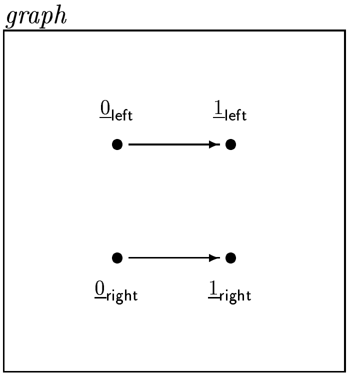
Visualizing ternary relations is difficult. The following definition enables partial visualizations.

Definition 27 (graph of a connector) $\quad$ For a connector f:
$E_f := \{\alpha\beta \mid \alpha\beta f \in R^Y\}$.
$\mathcal{G}_f := (\mathsf{Vert}, E_f)$ is called the graph of f. □

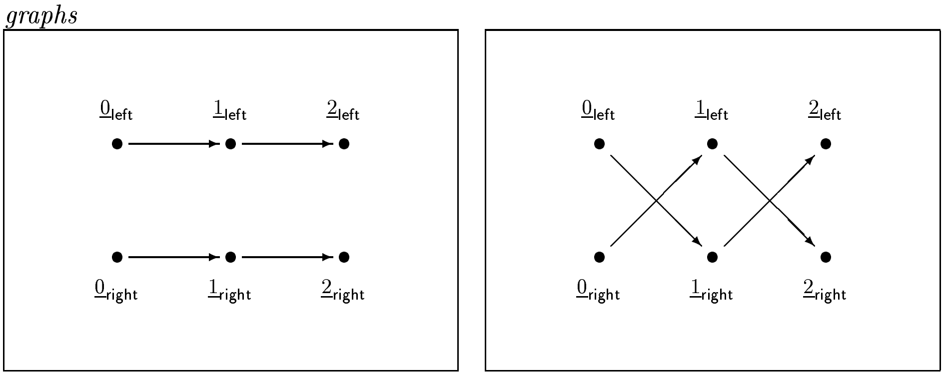
As mentioned before, the above definitions are relative to $Y = (\mathsf{Set}, <, \mathsf{Part})$. When referring to other assorted sequences, we use upper indices. For example if we speak about an assorted sequence U, this implies $U = (\mathsf{Set}^U, <^U, \mathsf{Part}^U)$, $\mathsf{N}^U = |\mathsf{Set}^U|$, $\underline{i}^U$ the i-th element of $(\mathsf{Set}^U, <^U)$ and so on. Slightly deviating from this convention, we write $\mathcal{U}$ instead of $\mathcal{Y}^U$ for the structure associated to U.

Definition 28 ($\mathsf{Simple}(n)$) $\quad \mathsf{Simple}(n)$ is the assorted sequence with $\mathsf{N}^{\mathsf{Simple}(n)} = n$ and
$\mathsf{Part}^{\mathsf{Simple}(n)} = \{\mathsf{Set}^{\mathsf{Simple}(n)}\}$. □

Example 29 *Let* $Y = \mathsf{Simple}(1)$. *Then* Conn *has one element with the following graph*



Example 30 *Let* $Y = \mathsf{Simple}(2)$. *Then* Conn *has two elements with the following graphs*



Let $Y = (\mathsf{Set}, <, \mathsf{Part})$ be an arbitrary assorted sequence again.

Proposition 31 *For all* $\alpha \in \mathsf{Vert}$*,* $\{\alpha, \mathsf{mirror}(\alpha)\}$ *is invariant under* $\mathsf{Aut}(\mathcal{Y})$*.*

Proposition 32 $\mathsf{Aut}(\mathcal{Y}^{\|}) = \mathsf{Aut}(\mathcal{Y}^{\times}) \subset \mathsf{Aut}(\mathcal{Y})$

Proposition 33 *For all* $M \in \mathsf{Part}$*,* Conn^M *is invariant under* $\mathsf{Aut}(\mathcal{Y})$*.*

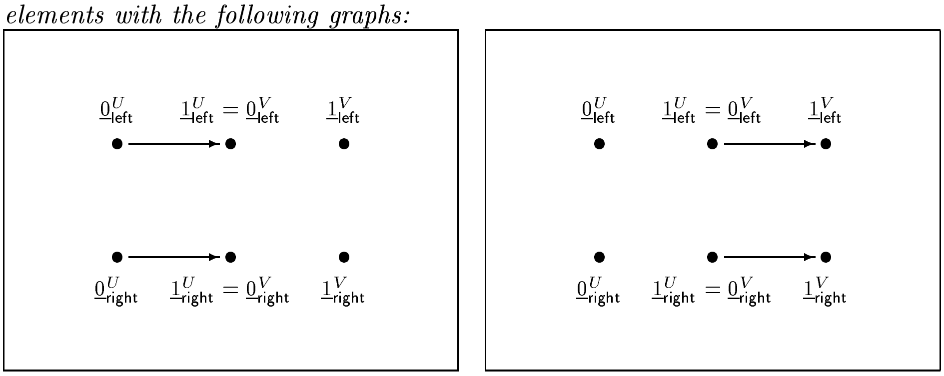
Proposition 34 *For all* $M \in \mathsf{Part}$*,* $\mathsf{Aut}(\mathcal{Y})$ *operates transitive on* Conn^M *(i.e. for all* $f, g \in \mathsf{Conn}^M$*, there is an automorphism of* $\mathcal{Y}$ *that maps* f *to* g*).*

Definition 35 ($U + V$) Let U, V be assorted sequences. Then $U + V$ is the assorted sequence with $\mathsf{Set}^{U+V} := \mathsf{Set}^U \cup \mathsf{Set}^V$, $\mathsf{Part}^{U+V} = \mathsf{Part}^U \cup \mathsf{Part}^V$ and $\prec := <^{U+V}$ defined by $\underline{0}^U \prec \ldots \prec \underline{\mathsf{N}-1}^U \prec \underline{0}^V \prec \ldots \prec \underline{\mathsf{N}-1}^V$ □

Convention 36 *When we write '*$U + V$*', we tacitly assume* $\underline{\mathsf{N}}^U = \underline{0}^V$

Proposition 37 *With the above convention we have for* $W = U + V$*:*
$\mathsf{Univ}(\mathcal{W}) = \mathsf{Univ}(\mathcal{U}) \cup \mathsf{Univ}(\mathcal{V})$
$R^{\mathcal{W}} = R^{\mathcal{U}} \cup R^{\mathcal{V}}$

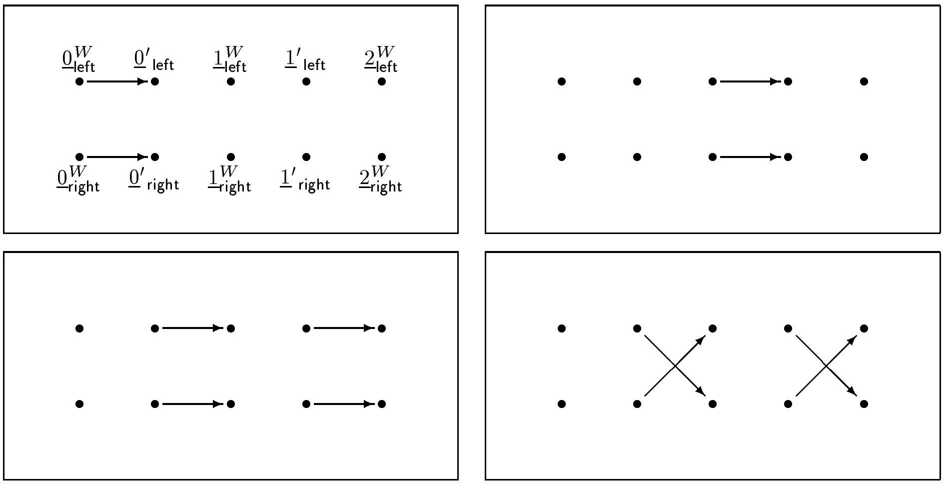
Example 38 *Let* U, V *be copies of* $\mathsf{Simple}(1)$*,* $W := U + V$*. Then* Conn^W *has two elements with the following graphs:*



Definition 39 ($\mathsf{Zip}(Y)$) Let V be an assorted sequence. $\mathsf{Zip}(V)$ is the assorted sequence with $\mathsf{Set}^{\mathsf{Zip}(V)} := \mathsf{Set}^V \cup \mathsf{Inlay}$, where $\mathsf{Inlay} := \{\underline{0}', \underline{1}', \ldots, \underline{\mathsf{N}-1}'\}$, $\mathsf{Part}^{\mathsf{Zip}(V)} := \mathsf{Part}^V \cup \{\mathsf{Inlay}\}$ and $\prec := <^{\mathsf{Zip}(V)}$ defined by $\underline{0}^V \prec \underline{0}' \prec \underline{1}^V \prec \underline{1}' \prec \ldots \prec \underline{\mathsf{N}-1}^V \prec \underline{\mathsf{N}-1}'$ □

Convention 40 *We assume* $\underline{\mathsf{N}}^{\mathsf{Zip}(V)} = \underline{\mathsf{N}}^V$*.*

Example 41 *Let W be as in example 38, $Z := \mathsf{Zip}(W)$, and Inlay as in definition 39. Then Conn^Z has four elements with the following graphs:*



Proposition 42 *Let V be an assorted sequence and $U := \mathsf{Zip}(V)$. Let Inlay be as in definition 39. Assume that f and g are partial automorphisms of $\mathcal{U}$ with*

- $\mathrm{def}(f) = \mathrm{def}(g) =: \mathrm{def}$,
- $\mathrm{def} \cap \mathsf{Conn}^{\mathsf{Inlay}} = \emptyset$,
- *for all* $\alpha \in \mathrm{def} \cap \mathsf{Vert}^U$, $f(\alpha), g(\alpha) \in \{\alpha, \mathsf{mirror}(\alpha)\}$
- *for all* $\alpha \in \mathrm{def} \cap \mathsf{Conn}^V$, $f(\alpha) = g(\alpha)$.

Then there is an automorphism ρ of $\mathcal{U}$ for which Conn^V is fix under ρ and $f = \rho \circ g$.

5 The Structures

For the rest of the paper, fix an arbitrary $k \in I\!N$ To prove the Hierarchy Theorem, we will define assorted sequences V_n (depending on k) and σ_0-formulas ν_n (not depending on k) such that ${V_n}^{\|} \models \nu_n$, ${V_n}^{\times} \not\models \nu_n$ and the duplicator wins $\mathrm{G}_0(n, k, {\mathcal{V}_n}^{\|}, {\mathcal{V}_n}^{\times})$. U_n and υ_n are intermediate stages in the definition of V_n and ν_n.

Definition 43 (U_n, V_n) U_n and V_n are the assorted sequences simultaneously defined by

- $U_0 := \mathsf{Simple}(1)$,
- For $n \geq 0$: $V_n := U_n^1 + D^1 + U_n^2 + D^2 + \ldots + U_n^{2^k} + D^{2^k}$, where U_n^i is a copy of U_n and D^i is a copy of $\mathsf{Simple}(2)$ for all $1 \leq i \leq 2^k$,
- For $n > 0$: $U_n := \mathsf{Zip}(V_{n-1})$.

□

6 The Formulas

In all formulas in this section, the variables x_1 and x_2 are free (among possibly others). We just write φ instead of $\varphi(x_1, x_2)$. As usual, when for example substituting x_1 and x_2 by y and z we write $\varphi(y, z)$.

In the sequel we use the following abbreviations:

$$\begin{array}{lll} R_f & := & Rx_1x_2f \\ \varphi + \psi & := & \exists z\varphi(x_1,z) \wedge \psi(z,x_2) \\ [\mathrm{DTC}\ \ \varphi] & := & [\mathrm{DTC}_{yz}\ \ \varphi(y,z)]x_1x_2 \end{array}$$

Definition 44 (sequential) φ is *sequential* if it can be derived by the following calculus:
$\frac{}{R_f}$; $\frac{\varphi,\psi}{(\varphi+\psi)}$; $\frac{\varphi}{\exists f\varphi}$; $\frac{\varphi}{[\mathrm{DTC}\ \ \varphi]}$. □

Proposition 45 *Suppose that ψ it a sequential formula for which $\mathcal{Y} \models \psi(\alpha,\beta)$ holds.*
Then there is an '$\exists fR_f$'-path from α to β.

Definition 46 (δ) $\delta := \exists f(R_f + R_f)$ □

Proposition 47 $Y = \mathsf{Simple}(2) \Leftrightarrow \mathcal{Y}^{\|} \models \delta(a,b)$

Proposition 48 *Suppose $U = \mathsf{Zip}(Y)$. Let Inlay be as in definition 39, $\mathbb{1}$ be the connector for Inlay that maps all $\underline{i}' \in \mathsf{Inlay}$ to $\|$. Let $\psi(\bar{v})$ be a sequential formula, $\bar{v}$ being all free variables of ψ, including x_1 and x_2. Then for all $\bar{\alpha} \in \mathcal{Y}$ we have*

$$\mathcal{Y} \models \psi(\bar{\alpha}) \Leftrightarrow \mathcal{U} \models \psi[\frac{R_f}{R_f + R_{\mathbb{1}}}](\bar{\alpha})$$

Here $\psi[\frac{R_f}{R_f+R_{\mathbb{1}}}]$ means the sequential formula ψ with all occurrences of R_f replaced by $R_f + R_{\mathbb{1}}$

Proposition 49 *Let $U, Y, \mathsf{Inlay}, \mathbb{1}$ as in proposition 48. Let $\psi(x_1,x_2)$ be a sequential formula with no other free variables than x_1 and x_2. Then*

$$\begin{array}{rcl} \mathcal{Y}^{\|} \models \psi(a,b) & \Leftrightarrow & \mathcal{U}^{\|} \models \psi[\frac{R_f}{R_f+R_{\mathbb{1}}}](a,b) \\ & \Leftrightarrow & \mathcal{U}^{\|} \models \psi[\frac{R_f}{R_f+R_\alpha}](a,b) \quad (\textit{ for all } \alpha \in \mathsf{Conn}^{\mathsf{Inlay}}) \\ & \Leftrightarrow & \mathcal{U}^{\|} \models \exists g\psi[\frac{R_f}{R_f+R_g}](a,b) \end{array}$$

The same holds for $\times$ instead of $\|$.

(Take notice of convention 40. For the second equivalence use proposition 34.)

Definition 50 (υ_n, ν_n) υ_n and ν_n are the sequential formulas simultaneously defined by

- $\upsilon_0 := \exists fR_f$
- For $n \geq 0$: $\nu_n := [\mathrm{DTC}\ \ (\upsilon_n + \delta)]$
- For $n > 0$: $\upsilon_n := \exists g\nu_{n-1}[\frac{R_f}{R_f+R_g}]$

□

Proposition 51 $\mathrm{Qr}(\nu_n) = \mathrm{Qr}(\upsilon_n) + 1 = n + 1$

Theorem 52

(i) $\mathcal{U}_n{}^{\|} \models \upsilon_n(a,b), \quad \mathcal{U}_n{}^{\times} \not\models \upsilon_n(a,b)$

(ii) $\mathcal{V}_n{}^{\|} \models \nu_n(a,b), \quad \mathcal{V}_n{}^{\times}(Y_n) \not\models \nu_n(a,b)$

Proof: [By simultaneous induction on n]

(i) For $n = 0$**:** obvious.

(ii) For $n \geq 0$**:** Let U^i_n, D^i be as in definition 43. As in proposition 37 regard $\mathcal{V}_n$ as the concatenation of the structures $\mathcal{U}^1_n, \mathcal{D}^1, \ldots, \mathcal{U}^{2^k}_n, \mathcal{D}^{2^k}$. Remember that by convention 36 for $s \in \{\mathsf{left}, \mathsf{right}\}$: $\underline{\mathsf{N}}^{U^i_n}_s = \underline{0}^{D^i}_s$ and $\underline{\mathsf{N}}^{D^i}_s = \underline{0}^{U^{i+1}_n}_s$.

Recall that $\nu_n := [\mathrm{DTC}\ \ \upsilon_n + \delta]$. We have to show that there is a deterministic $\upsilon_n + \delta$-path from $\underline{0}^{U^1_n}_{\mathsf{right}}$ to $\underline{\mathsf{N}}^{D^{2^k}}_{\mathsf{right}}$, but not from $\underline{0}^{U^1_n}_{\mathsf{right}}$ to $\underline{\mathsf{N}}^{D^{2^k}}_{\mathsf{left}}$. For both it is sufficient to show

CLAIM: Let $\alpha = \underline{0}^{U^i_n}_{\mathsf{right}}$. Then $\quad \mathcal{V}_n \models (\upsilon_n + \delta)(\alpha, \beta) \quad \Leftrightarrow \quad \beta = \underline{\mathsf{N}}^{D^i}_{\mathsf{right}}$.

<u>"$\Leftarrow$":</u> easy.

<u>"$\Rightarrow$":</u> By definition, there is an element $\gamma \in \mathcal{V}_n$ such that $\mathcal{V}_n \models \upsilon_n(\alpha, \gamma)$ and $\mathcal{V}_n \models \delta(\gamma, \beta)$. There is an element $g \in \mathsf{Conn}^{\mathcal{V}_n}$ such that there is a '$\exists f(R_f + R_g)$'-path from α to γ (apply theorem 45 to ν_{n-1} and recall $\upsilon_n := \exists g \nu_{n-1}[\frac{R_f}{R_f + R_g}]$). Clearly, $g \in \mathcal{U}^i_n$ and hence $\gamma \in \mathsf{Vert}^{U^i_n}$. From $\mathcal{V}_n \models \delta(\gamma, \beta)$ we see that $\gamma \in \{\underline{0}^{D^j}_{\mathsf{right}}, \underline{0}^{D^j}_{\mathsf{left}}\}$ for some j (remember $U_n = \mathsf{Zip}(V_{n-1})$ to confirm that there is no other possibility). We have $j = i$, because otherwise $\gamma \notin \mathcal{U}^i_n$. In fact, $\gamma = \underline{0}^{D^i}_{\mathsf{right}}$ and hence $\beta = \underline{\mathsf{N}}^{D^i}_{\mathsf{right}}$. $\gamma = \underline{0}^{D^i}_{\mathsf{left}} = \underline{\mathsf{N}}^{U^i_n}_{\mathsf{left}}$ would contradict (i), because we started with $\mathcal{V}_n \models \upsilon_n(\alpha, \gamma)$.

(i) For $n > 0$**:** Follows from (ii) for $n - 1$ by proposition 49.

$\blacksquare$

7 Playing the Game

The following definitions are for $m \in \mathbb{N} \cup \{*\}$

Definition 53 ($\mathrm{G}_m(n, k, Y)$) $\quad \mathrm{G}_m(n, k, Y) := \mathrm{G}_m(n, k, \mathcal{Y}^{\parallel}, \mathcal{Y}^{\times})$ $\square$

Note that $\mathcal{Y}^{\parallel}$ and $\mathcal{Y}^{\times}$ share their universe and thus $\mathrm{G}_m(n, k, Y)$ actually is a game played on one structure.

Definition 54 (neat) A situation of $\mathrm{G}_m(n, k, Y)$ is *neat*, if the couples (in the sense of definition 12) form a partial isomorphism and for all couples (α, β) either $\alpha \in \mathsf{Conn}$ and $\beta = \alpha$ or $\alpha \in \mathsf{Vert}$ and $\beta \in \{\alpha, \mathsf{mirror}(\alpha)\}$. $\square$

Definition 55 (isomorphic situations) Two situations $\mathfrak{S}$ and $\mathfrak{T}$ of $\mathrm{G}_m(n, k, Y)$ are *isomorphic* if there is an automorphism ρ of $Y^{\parallel}$ such that $\mathfrak{S}$ can be converted to $\mathfrak{T}$ by relocating all pebbles Q_i (but not the P_i) according to ρ. $\square$

Definition 56 (neatly wins) The duplicator *neatly wins* $\mathrm{G}_m(n, k, Y)$ if she has a strategy Str, such that any situation that can occur if she uses Str is isomorphic to a neat situation. Then Str is called a *neat strategy.* $\square$

Proposition 57 *If the duplicator has a neat strategy in a situation $\mathfrak{S}$ and $\mathfrak{S}$ is isomorphic to $\mathfrak{T}$, then she has also a neat strategy in the situation $\mathfrak{T}$.*

Proposition 58 *Propositions 15 and 16 still hold if we replace "wins" by "neatly wins".*

Definition 59 (Q-move) A Q-move is an $\exists$-move or an $\forall$-move. $\square$

Definition 60 (to copy, to mirror) Let in the game $\mathrm{G}_m(n, k, Y)$ the duplicator make a Q-move to some $\alpha \in \mathcal{Y}$. Then we say that the duplicator *copies* the move, if she places the corresponding pebble to α as well. On the other hand, if $\alpha \in \mathsf{Vert}$ and the duplicator places the corresponding pebble to *mirror*, then we say that she *mirrors* the move.
For $\alpha \in \mathsf{Vert}$, a couple (α, β) is called *even*, if $\alpha = \beta$ and *odd*, if $\alpha = \mathsf{mirror}(\beta)$. $\square$

Proposition 61 *Let V be an assorted sequence, $U := \mathsf{Zip}(V)$, Inlay be as in definition 39. Let $\mathfrak{S}$ and $\mathfrak{T}$ be neat situations of $\mathrm{G}_m(n, k, Y)$ for which*

- *the depth of the game is 0,*
- *no pebbles are on $\mathsf{Conn}^{\mathsf{Inlay}}$.*

Let all pebbles P_i in $\mathfrak{S}$ be at the same place as in $\mathfrak{T}$. Then $\mathfrak{S}$ and $\mathfrak{T}$ are isomorphic. (Use proposition 42.)

Theorem 62 *For $n \in \mathbb{N}$:*

(i) *The duplicator neatly wins $\mathrm{G}_*(n, k, U_n)$.*

(ii) *The duplicator neatly wins $\mathrm{G}_0(n, k, V_n)$.*

Proof:

(i) for $n = 0$: By definition the duplicator wins in the start situation, which is neat.

(ii) for $n \geq 0$: Let U_n^i, D^i be as in definition 43. In the initial situation we have precisely two couples, the even couple $(a^{\mathcal{Y}^{\parallel}}, a^{\mathcal{Y}^{\times}})$ and the odd couple $(b^{\mathcal{Y}^{\parallel}}, b^{\mathcal{Y}^{\times}})$. Let us tell the duplicator how to win $\mathrm{G}_0(n, k, V_n)$ neatly:

(a) Before the first I-move: Copy every Q-move to a connector.
Copy every Q-move to a vertex, unless the pebbled vertex is closer to some odd couple than to any even couple. In that case mirror the move.

Check that when the first I-move is done, there is an $l, 1 \leq l \leq 2^k$, such that

- there are no couples on $\mathcal{U}_n^l \setminus \{\underline{0}_{\mathsf{left}}^{U_n^l}, \underline{0}_{\mathsf{right}}^{U_n^l}, \underline{\mathsf{N}}_{\mathsf{left}}^{U_n^l}, \underline{\mathsf{N}}_{\mathsf{right}}^{U_n^l}\}$,
- for all $i < l$ all couples on $\mathcal{U}_n^i$ and $\mathcal{D}^i$ are even
- for all $i > l$ all couples on $\mathcal{U}_n^i$ and $\mathcal{D}^i$ are odd.

Note that no R-move back to depth 0 is possible. Thus on U_n^l, we are in the same situation as in $\mathrm{G}_*(n, k, U_n)$ when the obligatory initial I-move is done. By (i) there is a neat strategy Str for that game.

(b) After the first I-move: Copy every Q-move to $\mathcal{U}_n^i, i < l$.
Mirror every Q-move to $\mathcal{U}_n^i, i > l$.
Answer all moves to $\mathcal{U}_n^l$ according to Str.
In every situation, the automorphism of ${\mathcal{U}_n^l}^{\parallel}$ that makes the situation isomorphic to a neat situation, can be extended by identity to an automorphism of ${\mathcal{V}_n}^{\parallel}$.

(i) for $n > 0$: Remember $U_n = \mathsf{Zip}(V_{n-1})$. Let Inlay be as in definition 39. By propositions 16 and 58 it is sufficient to show that the duplicator neatly wins $\mathrm{G}_m(n, k, U_n)$ for every $m > 0$. Let m be given. By (ii) for $n-1$ and propositions 15 and 58 there is a neat strategy Str for $\mathrm{G}_{m-1}(n-1, k, V_{n-1})$. Note that $\mathrm{G}_m(m, \ldots)$ and $\mathrm{G}_{m-1}(n-1, \ldots)$ only differ in the number of couples that remain on the board in R-moves to depth 0.
Now we can advise the duplicator how to win $\mathrm{G}_m(n, k, Y)$ neatly:

(a) Before the first R-move to depth 0: Answer all Q-moves to elements $\mathcal{V}_{n-1}$ according to Str.
Copy all Q-moves to $\mathsf{Conn}^{\mathsf{Inlay}}$.
For a Q-move to some $\underline{i'}_s, \underline{i'} \in \mathsf{Inlay}$: Ask Str whether to copy or to mirror a Q-move to $\underline{i+1}^{V_{n-1}}{}_s$ and carry over Str's instruction to $\underline{i'}_s$.

(b) R-move to depth 0:

Case 1: At least one pair (P_i, Q_i) of pebbles remains on $\mathsf{Conn}^{\mathsf{Inlay}}$: Ignore that pair and proceed as in (a). We can ignore (P_i, Q_i) because they are both located on the same $\alpha \in \mathsf{Conn}^{\mathsf{Inlay}}$. This cannot interfere with the strategy in (a).

Case 2: All couples are removed from $\mathsf{Conn}^{\mathsf{Inlay}}$: Play a virtual game of $\mathrm{G}_m(n, k, U_n)$. Let the virtual challenger start with a series of $\exists$-moves and let him place the virtual P_i to where the real P_i are. Answer these virtual moves as in (a). By proposition 61, the resulting virtual situation is isomorphic to the real situation. By proposition 57, we can assume that the inducing automorphism equals identity and proceed as in (a).

■

Now the Hierarchy Theorem (theorem 7) follows from the Game Theorem (theorem 13), theorem 52,(ii) and theorem 62,(ii).

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